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Minimal distortion morphs generated by time-dependent vector fields

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ABSTRACT

A morph between two Riemannian n -manifolds is an isotopy between them together with the set of all intermediate manifolds equipped with Riemannian metrics. We propose measures of the distortion produced by some classes of morphs and diffeomorphisms between two isotopic Riemannian n -manifolds and, with respect to these classes, prove the existence of minimal distortion morphs and diffeomorphisms. In particular, we consider the class of time-dependent vector fields (on an open subset Ω of \mathbb{R}^{n+1} in which the manifolds are embedded) that generate morphs between two manifolds M and N via an evolution equation, define the bending and the morphing distortion energies for these morphs, and prove the existence of minimizers of the corresponding functionals in the set of time-dependent vector fields that generate morphs between M and N and are L^2 functions from $[0, 1]$ to the Sobolev space $W_0^{k,2}(\Omega, \mathbb{R}^{n+1})$.

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1. Introduction

1.1. Summary of results

Let M and N be compact and orientable smooth Riemannian n -manifolds isometrically embedded into \mathbb{R}^{n+1} . A morph between M and N is an isotopy between them together with the set of all intermediate manifolds equipped with the Riemannian metrics inherited from \mathbb{R}^{n+1} . Every morph or diffeomorphism between isotopic manifolds produces distortion via stretching and bending. We define functionals that measure distortion and prove the existence of minimal distortion morphs and diffeomorphisms.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set containing the manifolds M and N . We define functionals E and \mathcal{E} that measure the distortion of diffeomorphisms and morphs respectively generated by time-dependent vector fields $v : \Omega \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ via the evolution equation $dq/dt = v(q, t)$ and prove the existence of minimizers of E and \mathcal{E} in an admissible set \mathcal{A}_P^k of time-dependent vector fields, which is a subset of the closed ball of radius P in the Hilbert space \mathcal{H}^k of all L^2 functions from $[0, 1]$ to the Sobolev space $W_0^{k,2}(\Omega; \mathbb{R}^{n+1})$, where $k \in \mathbb{N}$.

We compute the Euler–Lagrange equations for the deformation energy functional defined on $\text{Diff}(M, N)$ and show that the radial map between a manifold and its rescaled version is a critical point of the deformation energy functional. We prove the existence of minimal deformation holomorphic diffeomorphisms of Riemann surfaces.

1.2. Background and motivation

A fundamental problem in Riemannian geometry and related areas is to determine whether two diffeomorphic compact Riemannian manifolds (M, g_M) and (N, g_N) are isometric; that is, if there exists a diffeomorphism $h : M \rightarrow N$ such that

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$h^*g_N - g_M = 0$, where h^*g_N denotes the pull-back of g_N by h . If no such diffeomorphism exists, it is important to know whether there exists a diffeomorphism that most closely resembles an isometry. This is accomplished by minimization of the deformation energy functional

$$\Phi(h) = \int_M \|h^*g_N - g_M\|^2 \omega_M, \quad (1)$$

over the space $\text{Diff}(M, N)$ of diffeomorphisms between M and N , where ω_M is the volume form on M and $\|\cdot\|$ is the fiber norm on the bundle of all $(0, 2)$ -tensor fields on M generated by the fiber metric $g_M^* \otimes g_M^*$ (see [5], p. 63).

The minimization problem takes on added significance once the physical interpretation of the tensor $h^*g_N - g_M$ is recognized: it is exactly the (nonlinear) strain tensor corresponding to the deformation h in case g_M and g_N are Riemannian metrics inherited from Euclidean space. Thus, this functional and its variants must occur in physical problems. Indeed, the minimal distortion problem arises, for example, in manufacturing, computer graphics, movie making, and medical imaging.

The problem of bending a sheet of metal to a desired shape using minimal energy has been studied (see [21,4]); but, algorithms for numerical approximations are proposed and used without proving the existence of minimizers.

An animation might require an aesthetically pleasing transformation that takes one image to another through intermediate shapes. Such a transformation is called a morph, or a metamorphosis (see [20] for a survey on morphing). A desirable morph might be defined as a minimizer of a cost functional that measures the distortion energy. In [17], the distortion energy of an elastically deformable surface $r: U \times [0, T] \rightarrow \mathbb{R}^3$, where $U \subset \mathbb{R}^3$ is an open set, produced by a deformation $h = r(\cdot, T)$ is defined as the integral of weighted norms of the local coordinate representations of the strain tensor $h^*g_T - g_0$ and the tensor $h^*\Pi_T - \Pi_0$, where g_t and Π_t are the first and the second fundamental forms of the surface $r(U, t)$ at the time $t \in [0, T]$.

Image matching and image registration is an important subject in medical imaging. Image matching is used for determining the existence of abnormalities (distortions due to underlying medical conditions) in two images, taken at different times, of the same organism. The problem of registration of a population of images to one template for the purpose of statistical analysis is another instance of image registration. One approach to the image matching problem is by minimization of a distortion functional (see [7,9,11,18]).

In the following sections, we will discuss the underlying mathematical problem of the existence of minimal distortion diffeomorphisms and morphs between embedded manifolds of codimension one.

1.3. Mathematical preliminaries, definitions, and results

Given a smooth oriented n -manifold S (perhaps with boundary) isometrically embedded into \mathbb{R}^{n+1} , we let g_S , ω_S , and Π_S denote the Riemannian metric, volume form, and second fundamental form on S associated with this embedding. Also, we let $\text{Int } S$ (respectively, ∂S) denote the interior (respectively, the boundary) of the manifold S .

Definition 1.1. Let M and N be isotopic compact connected smooth n -manifolds (perhaps with boundary) embedded in \mathbb{R}^{n+1} such that M is oriented. A C^∞ isotopy $F: M \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ together with all the intermediate manifolds $M^t := F(M, t)$, equipped with the orientations induced by the maps $f^t = F(\cdot, t): M \rightarrow M^t$ and the Riemannian metrics g_t inherited from \mathbb{R}^{n+1} , is called a (smooth) *morph* from M to N .

We denote the set of all smooth (respectively, C^r) diffeomorphisms between manifolds M and N by $\text{Diff}(M, N)$ (respectively, $\text{Diff}^r(M, N)$). Similarly, we denote the set of all smooth morphs between M and N by $\mathcal{M}(M, N)$. If F is an isotopy, then each map $F(\cdot, t): M \rightarrow M^t$ induces smooth diffeomorphisms $\text{Int } F(\cdot, t): \text{Int } M \rightarrow \text{Int } M^t$ and $\partial F(\cdot, t): \partial M \rightarrow \partial M^t$ by restriction.

In addition, we consider morphs between manifolds M and N with different regularity properties. For example, we let $\mathcal{M}^{r,ac}(M, N)$ denote the set of all continuous isotopies $F: M \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ between M and N such that for each $p \in M$ the map $t \mapsto F(p, t)$ is absolutely continuous on $[0, 1]$ and for each $t \in [0, 1]$ the function $p \mapsto F(p, t)$ is a C^r diffeomorphism from M onto its image. As in the case of smooth morphs, the diffeomorphism $F(\cdot, t): M \rightarrow M^t$ induces an orientation on the intermediate manifold M^t .

There are several choices for cost functionals that measure the distortion of a diffeomorphism $h \in \text{Diff}(M, N)$ or a morph $F \in \mathcal{M}(M, N)$.

A complete theory of the existence of minimizers of cost functionals that measure distortion of diffeomorphisms and morphs due to change of volume is presented in [6]. In this case, the value of the distortion energy functional at a diffeomorphism $h: M \rightarrow N$ is defined to be the square of the infinitesimal relative change of volume $|J(h)| - 1$ produced by h integrated over the manifold M , where $J(h)$ is the Jacobian determinant of h . This functional does not take into account the distortion of shape produced by h , which is captured by functional (1) and functional (2) below.

The general problem of the existence of minimizers of Φ is open. The special case where M and N are one-dimensional is studied in [3] where, among other results, the functional Φ is shown to have no minimizer in case M and N are circles with the radius of N smaller than the radius of M . Thus, a solution of the general problem must take into account at

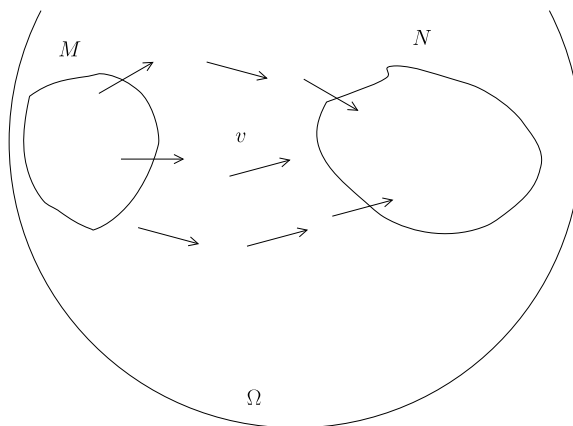


Fig. 1. The time-dependent vector field $v : \Omega \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ generates the morph $F^v(p, t)$, which is the solution of the initial value problem $dq/dt = v(q, t)$, $q(0) = p$.

least some global properties of the metric structures of the manifolds M and N . On the other hand, we will prove the existence of minimizers in case M and N are Riemann spheres or compact Riemann surfaces of genus greater than one and the admissible set is $\text{HD}(M, N) = \{h \in \text{Diff}(M, N) : h \text{ is a holomorphic map}\}$. More precisely, the following theorem will be proved in Section 3.

Theorem 1.2.

- (i) Let $h_R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the radial map given by $h_R(p) = Rp$ for some number $R > 0$. If $M = \mathbb{S}^2$ is the 2-dimensional unit sphere isometrically embedded into \mathbb{R}^3 and $N = h_R(M)$, then $h \in \text{HD}(M, N)$ is a global minimizer of the functional Φ , restricted to the admissible set $\text{HD}(M, N)$, if and only if $h = f \circ h_R|_M$, where f is an isometry of N .
- (ii) Let M and N be compact Riemann surfaces. If $\text{HD}(M, N)$ is not empty and the genus of M is at least two, then there exists a minimizer of the functional Φ in $\text{HD}(M, N)$.

We note that the diffeomorphism $h = f \circ h_R|_M$ in the latter theorem satisfies the Euler–Lagrange equations for the functional Φ with its natural domain $\text{Diff}(M, N)$ (see Appendix A). The Euler–Lagrange equations for the deformation energy functional $\Phi : \text{Diff}(M, N) \rightarrow \mathbb{R}_+$ are highly nonlinear and rather complicated (see Proposition A.3), which discouraged us from using them to show the existence of minimizers of the functional Φ on $\text{Diff}(M, N)$.

If we wish to match, in addition to the Riemannian metrics, the embeddings of the manifolds M and N (to avoid, for example, zero distortion energy maps between a square and a round cylinder in \mathbb{R}^3), we arrive at the problem of minimization of the functional

$$\Lambda(h) := \int_M \|h^* g_N - g_M\|^2 \omega_M + \int_M \|h^* \Pi_N - \Pi_M\|^2 \omega_M \quad (2)$$

over the space of diffeomorphisms between M and N , where Π_M and Π_N are the second fundamental forms on the manifolds M and N .

One of the difficulties encountered in attempts to minimize Φ over $\text{Diff}(M, N)$ is the lack of a complete understanding of the structure of this infinite-dimensional space. The natural new approach is to linearize; that is, replace $\text{Diff}(M, N)$ with a subset of a linear function space. Using this approach, which already appears in the literature on image deformation (see [7, 9, 11, 18]), we define our distortion energy functionals on time-dependent vector fields that generate morphs (see Fig. 1).

Let us denote the Euclidean norm of an element $A \in \mathbb{R}^m$ by $|A|$ or by $|A|_{\mathbb{R}^m}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open ball containing the manifolds M and N , $C_c^\infty(\Omega; \mathbb{R}^{n+1})$ the space of all smooth functions from Ω to \mathbb{R}^{n+1} with compact support, and $V^k := W_0^{k,2}(\Omega; \mathbb{R}^{n+1})$ the closure of $C_c^\infty(\Omega; \mathbb{R}^{n+1})$ in the Sobolev space $W^{k,2}(\Omega; \mathbb{R}^{n+1})$ (see [10]).

The space V^k is a Hilbert space with the inner product

$$\langle f, g \rangle_{V^k} = \sum_{i=1}^{n+1} \sum_{|\alpha| \leq k} \int_{\Omega} D^\alpha f_i D^\alpha g_i dx,$$

where $f = (f_1, \dots, f_{n+1}) : \Omega \rightarrow \mathbb{R}^{n+1}$, $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ is a multi-index with nonnegative integer components, $|\alpha| = \alpha_1 + \dots + \alpha_{n+1}$, and $D^\alpha f_i = \partial^{|\alpha|} f_i / \partial x_1^{\alpha_1} \dots \partial x_{n+1}^{\alpha_{n+1}}$ is the corresponding weak partial derivative of f_i . We choose $k \in \mathbb{N}$ large

enough so that the Sobolev space $W_0^{k,2}(\Omega)$ is embedded into $C^r(\bar{\Omega})$ and $r \geq 1$. By the Sobolev Embedding Theorem (see [2, 10]), it suffices to choose $k \geq (n+1)/2 + r + 1$.

Consider time-dependent vector fields $v : \Omega \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ on Ω that belong to the Hilbert space

$$\mathcal{H}^k = L^2(0, 1; V^k) \quad (3)$$

(see Fig. 1). By an abuse of notation, we will sometimes write $v(x, t) = v(t)(x)$ for $v \in \mathcal{H}^k$ and $(x, t) \in \Omega \times [0, 1]$. A time-dependent vector field $v : \Omega \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ belongs to the Hilbert space \mathcal{H}^k if its norm $\|v\|_{\mathcal{H}^k} := (\int_0^1 \|v(\cdot, t)\|_{V^k}^2 dt)^{\frac{1}{2}}$ is finite. The inner product on \mathcal{H}^k is defined by

$$\langle v, w \rangle_{\mathcal{H}^k} = \int_0^1 \langle v(\cdot, t), w(\cdot, t) \rangle_{V^k} dt.$$

Every vector field $v \in \mathcal{H}^k$ generates a morph $F^v : M \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ from M to $F^v(M, 1)$ via the evolution equation

$$\frac{dq}{dt} = v(q, t). \quad (4)$$

More precisely, let $\eta^v(t; t_0, x)$ be the evolution operator [8] of Eq. (4); that is, for every $t_0 \in [0, 1]$ and $x \in \Omega$ the function $t \mapsto \eta^v(t; t_0, x)$ solves Eq. (4) and satisfies the initial condition $\eta^v(t_0; t_0, x) = x$. The morph F^v is defined by $F^v(p, t) = \eta^v(t; 0, p)$ for all $(p, t) \in M \times [0, 1]$. By the properties of the evolution operator η^v , which have been studied in [9] and [19], the morph $F^v(p, t)$ is of class $\mathcal{M}^{r,ac}(M, F^v(M, 1))$ (see Lemmas B.1 and B.2). The time-one map of the evolution operator η^v is defined to be $\phi^v(x) := \eta^v(1; 0, x)$ for all $x \in \Omega$, and we define $\psi^v = \phi^v|_M$.

Let \mathcal{A}_P^k be the admissible set of all time-dependent vector fields in \mathcal{H}^k that generate morphs between the manifolds M and N and are bounded by a uniform positive constant P . In symbols,

$$\mathcal{A}_P^k = \{v \in \mathcal{H}^k : \psi^v \in \text{Diff}^r(M, N) \text{ and } \|v\|_{\mathcal{H}^k} \leq P\}. \quad (5)$$

We will prove that for P sufficiently large, the admissible set \mathcal{A}_P^k is nonempty and \mathcal{A}_P^k is weakly closed in \mathcal{H}^k (see Lemma 2.6).

Let $\mathcal{T}_s^r(M)$ denote the set of all continuous tensor fields on M contravariant of order r and covariant of order s (also called type (r, s)). For a tensor field $\tau_N \in \mathcal{T}_s^0(N)$ and a diffeomorphism $h : M \rightarrow N$, $h^*\tau_N$ denotes the pull-back of τ_N to M .

For each $t \in [0, 1]$ and $v \in \mathcal{A}_P^k$, the manifold $M^{v,t} := F^v(M, t)$ is called an intermediate state of the morph F^v between manifolds M and N generated by the time-dependent vector field v . We endow this intermediate state with the Riemannian metric g_t^v inherited from its embedding in \mathbb{R}^{n+1} and let Π_t^v denote the corresponding second fundamental form.

Definition 1.3. Let B_1 and B_2 be nonnegative real numbers, F^v the morph, and ϕ^v the time-one map generated by the time-dependent vector field $v \in \mathcal{A}_P^k \subset \mathcal{H}^k$ via the evolution equation (4). Recall that $\psi^v := \phi^v|_M$. The *bending distortion energy* of v is

$$E(v) = E(v; B_1, B_2) = B_1 \int_M \|(\psi^v)^* g_N - g_M\|^2 \omega_M + B_2 \int_M \|(\psi^v)^* \Pi_N - \Pi_M\|^2 \omega_M$$

and the *morphing distortion energy* of v is

$$\mathcal{E}(v) = \mathcal{E}(v; B_1, B_2) = B_1 \int_0^1 \int_M \|F^v(\cdot, t)^* g_t^v - g_M\|^2 \omega_M dt + B_2 \int_0^1 \int_M \|F^v(\cdot, t)^* \Pi_t^v - \Pi_M\|^2 \omega_M dt,$$

where $\|\cdot\|$ is the fiber norm on the tensor bundle $\mathcal{T}_2^0(M)$ generated by the fiber inner product $g_M^* \otimes g_M^*$. (Note: We will use the same notation for the fiber norm on the tensor bundle $\mathcal{T}_s^0(M)$ generated by the inner product $\bigotimes_{i=1}^s g_M^*$.)

We will prove that the functionals E and \mathcal{E} have minimizers in \mathcal{A}_P^k .

Theorem 1.4. If $P > 0$ and $k \in \mathbb{N}$ are sufficiently large, then each of the functionals $E : \mathcal{A}_P^k \rightarrow \mathbb{R}_+$ and $\mathcal{E} : \mathcal{A}_P^k \rightarrow \mathbb{R}_+$ has a minimizer in the admissible set \mathcal{A}_P^k .

The detailed conditions on the constants P and k are formulated in Theorem 2.8.

We note that each diffeomorphism $\psi^v : M \rightarrow N$ generated by a time-dependent vector field $v \in \mathcal{A}_P^k$ is isotopic, as a map from M to \mathbb{R}^{n+1} , to the inclusion map $i : M \rightarrow \mathbb{R}^{n+1}$ via the isotopy $F^v \in \mathcal{M}^{r,ac}(M, N)$. To minimize the distortion

energy of diffeomorphisms from other isotopy classes, we replace the map ψ^v in the definition of the functional E by the diffeomorphism $\psi^v \circ \phi : M \rightarrow N$, where ϕ is a fixed diffeomorphism on M . The existence of minimizers of the functional E with the above adjustment guarantees the existence of minimizers of the functionals Φ and Λ defined in displays (1) and (2) in a restricted admissible set of all C^2 diffeomorphisms between the manifolds M and N , which, considered as maps from M to \mathbb{R}^{n+1} , are isotopic to a given map $\phi : M \rightarrow \mathbb{R}^{n+1}$.

Theorem 1.5. *If $P > 0$ and $k \in \mathbb{N}$ are sufficiently large, then for every $\phi \in \text{Diff}(M)$ both functionals Φ and Λ defined in displays (1) and (2) respectively have minimizers in the admissible set*

$$\mathcal{B}_{P,\phi}^k := \{h \in \text{Diff}^2(M, N) : h = \psi^v \circ \phi \text{ for some } v \in \mathcal{A}_P^k\}.$$

2. Bending and morphing via time-dependent vector fields in \mathbb{R}^{n+1}

In this section we prove Theorem 1.4.

We will show that the admissible set \mathcal{A}_P^k is nonempty if P is sufficiently large.

Lemma 2.1. *Let M and N be manifolds as in Definition 1.1. Let F be a smooth morph between the manifolds M and N and assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open ball in \mathbb{R}^{n+1} containing the image $F(M \times [0, 1])$ of the morph F . There exists $P_0 > 0$ such that the admissible set \mathcal{A}_P^k is nonempty whenever $P \geq P_0$ and $k \geq \frac{n+1}{2} + 2$.*

Proof. The morph $F \in \mathcal{M}(M, N)$ defines the \mathbb{R}^{n+1} valued function

$$v(y, t) = \frac{\partial F}{\partial t}([F(\cdot, t)]^{-1}(y), t)$$

on the compact subset $Q = \{(F(x, t), t) : (x, t) \in M \times [0, 1]\}$ of $\mathbb{R}^{n+1} \times \mathbb{R}$.

We will extend the function v to a smooth vector field $w \in \mathcal{H}^k$ such that $\psi^w(M) = N$.

First, notice that the smooth map $G : M \times [0, 1] \rightarrow \mathbb{R}^{n+1} \times \mathbb{R}$ defined by $G(x, t) = (F(x, t), t)$ is a proper map ($M \times [0, 1]$ is compact) and an injective immersion, hence an embedding (see [1]). Therefore, $Q = G(M \times [0, 1])$ is a submanifold (with boundary) of $\mathbb{R}^{n+1} \times \mathbb{R}$ (see [13]).

Next, notice that the map $G_1 : Q \rightarrow M \times [0, 1]$ defined by $G_1(y, t) = ([F(\cdot, t)]^{-1}(y), t)$ is the inverse of G . Because G is an immersion, hence a local diffeomorphism, the map G_1 is smooth. Therefore, the map $v : Q \rightarrow \mathbb{R}^{n+1}$ is smooth because it is the composition of two smooth maps $G_1 : Q \rightarrow M \times [0, 1]$ and $\frac{\partial F}{\partial t} : M \times [0, 1] \rightarrow \mathbb{R}^{n+1}$.

The smooth function $v : Q \rightarrow \mathbb{R}^{n+1}$ can be extended locally. That is, for every $(y, t) \in Q$ there exist an open set $U \subset \mathbb{R}^{n+1} \times \mathbb{R}$ such that $(y, t) \in U$ and a smooth function $v_1 : U \rightarrow \mathbb{R}^{n+1}$ such that $v_1|_{U \cap Q} = v|_{U \cap Q}$. This local extension property follows from a more general fact about smooth functions defined on submanifolds: Let S be an s -dimensional smooth submanifold (perhaps with boundary) of \mathbb{R}^m and let $f : S \rightarrow \mathbb{R}$ be a smooth function. Then for every $x \in S$ there exists an open set $W \subset \mathbb{R}^m$ containing x and a smooth function $f_1 : W \rightarrow \mathbb{R}$ such that $f|_{W \cap S} = f_1|_{W \cap S}$. It is easy to construct a local extension of the function f using submanifold charts on S and the definition of a smooth function whose domain is a submanifold with boundary. The details are left to the reader.

Therefore, the function $v : Q \rightarrow \mathbb{R}^{n+1}$ satisfies the conditions of the smooth Tietze extension theorem (see [1]) and can be extended from the closed set $Q \subset \mathbb{R}^{n+1} \times \mathbb{R}$ by a smooth map $\tilde{v} : \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$.

Finally, define $w(x, t) = \rho(x)\tilde{v}(x, t)$, where $\rho : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a smooth bump function such that $\rho \equiv 1$ on Q and $\rho \equiv 0$ on $\partial\Omega$, and set $P_0 := \|w\|_{\mathcal{H}^k}$. \square

From now on, we assume that the open set Ω in \mathbb{R}^{n+1} is chosen as in the latter lemma and the constant $P > 0$ is large enough so that the set \mathcal{A}_P^k is not empty; the number k of weak derivatives satisfies the inequality $k \geq (n+1)/2 + r + 1$, where $r \geq 1$.

For each $v \in \mathcal{A}_P^k$, the time-one map $\psi^v : M \rightarrow N$ transforms the interior (respectively, the boundary) of the manifold M to the interior (respectively, the boundary) of the manifold N . The existence and the convergence properties of the evolution operators generated by vector fields $v \in \mathcal{H}^k$ via the evolution equation (4) have been studied in [9,19]. For convenience of the reader, we state some of these properties (which will be useful in our proofs) in Appendix B.

As mentioned in Section 1.3, every time-dependent vector field $v \in \mathcal{A}_P^k \subset \mathcal{H}^k$ generates a morph F^v between the manifolds M and N of class $\mathcal{M}^{r,ac}(M, N)$.

Let us recall the distortion energy functionals $E : \mathcal{H}^k \rightarrow \mathbb{R}_+$ and $\mathcal{E} : \mathcal{H}^k \rightarrow \mathbb{R}_+$ (see Definition 1.3).

One of the main ingredients in the proof of Theorem 1.4 is the weak continuity of the functionals E and \mathcal{E} . We will prove the weak continuity of more general auxiliary functionals, where the tensor fields τ_M and τ_N in the following definition will later be replaced by the first and the second fundamental forms on the manifolds M and N respectively.

Definition 2.2. Let M and N be manifolds as in Definition 1.1. For given continuous tensor fields τ_M and τ_N of type $(0, s)$ on M and N respectively, the functional $J : \mathcal{A}_P^k \rightarrow \mathbb{R}_+$ is defined by

$$J(v) = \int_M \|(\psi^v)^* \tau_N - \tau_M\|^2 \omega_M.$$

Let $v \in \mathcal{A}_p^k$ be a time-dependent vector field that generates a morph $F^v \in \mathcal{M}^{r,ac}(M, N)$ from the manifold M to N . Recall that the intermediate state at the time $t \in [0, 1]$ of the morph F^v is denoted by $M^{v,t}$. The Riemannian metric and the second fundamental form on $M^{v,t}$, which are associated with the embedding of $M^{v,t}$ into \mathbb{R}^{n+1} , are denoted by g_t^v and Π_t^v respectively. The functionals $I_1 : \mathcal{A}_p^k \rightarrow \mathbb{R}_+$ and $I_2 : \mathcal{A}_p^k \rightarrow \mathbb{R}_+$ are given by

$$I_1(v) = \int_0^1 \int_M \|F^v(\cdot, t)^* g_t^v - g_M\|^2 \omega_M dt \quad (6)$$

and

$$I_2(v) = \int_0^1 \int_M \|F^v(\cdot, t)^* \Pi_t^v - \Pi_M\|^2 \omega_M dt. \quad (7)$$

Definition 2.3. Let X be an n -dimensional vector space equipped with the inner product g_X . Let $T_s^0(X) = \bigotimes_{i=1}^s X^*$. For every $v \in X$, we denote the norm of v with respect to the inner product g_X by $|v|_{g_X} = g_X(v, v)^{1/2}$ and the unit sphere by $S_{g_X} = \{v \in X : |v|_{g_X} = 1\}$. Define the norm on $T_s^0(X)$ by

$$\|b\|_{g_X} = \max_{v_i \in S_{g_X}} |b(v_1, \dots, v_s)|.$$

Another norm on $T_s^0(X)$ is defined by (see [15])

$$\|b\| = \bigotimes_{i=1}^s g_X^*(b, b).$$

Let $\mathbb{M}(X) \subset T_2^0(X)$ denote the metric space of all inner products on X with the metric $d(g, g') = \|g - g'\|_{g_X}$.

Note that if $\{e_1, \dots, e_n\}$ is an orthonormal basis of (X, g_X) , then

$$\|b\|^2 = \sum_{i_1, \dots, i_s=1}^n b(e_{i_1}, \dots, e_{i_s})^2. \quad (8)$$

Lemma 2.4. The function $\eta : T_s^0(X) \times \mathbb{M}(X) \rightarrow \mathbb{R}$ defined by $\eta(\beta, g) = \|\beta\|_g$ is continuous on its domain.

The proof of the latter lemma can be found in [5].

For a C^1 Riemannian manifold (S, g_S) , let $\|\cdot\|$ be the fiber norm on the bundle of all continuous tensor fields on S of type $(0, s)$ generated by the fiber inner product $\bigotimes_{i=1}^s g_S^*$.

Lemma 2.5. If b is a continuous tensor field of type $(0, s)$ on a C^1 Riemannian n -manifold (S, g_S) , then the function

$$z \mapsto \|b(z)\|_{g_S(z)} \quad (9)$$

is continuous on S . Moreover, the norms $\|\cdot\|$ and $\|\cdot\|_{g_S(z)}$ are uniformly equivalent; in fact,

$$\frac{1}{n^{s/2}} \|b(z)\| \leq \|b(z)\|_{g_S(z)} \leq \|b(z)\| \quad (10)$$

for all $b \in \mathcal{T}_s^0(S)$ and $z \in S$.

Proof. The continuity of the function defined in display (9) follows immediately from Lemma 2.4; and, the inequalities in display (10) can be easily derived from the definitions of the norms $\|\cdot\|$ and $\|\cdot\|_{g_S(z)}$ and formula (8). \square

Recall that a sequence $\{v^l\}_{l=1}^\infty \subset \mathcal{H}^k$ converges weakly to $v \in \mathcal{H}^k$ (in symbols, $v^l \rightharpoonup v$) as $l \rightarrow \infty$ if $\langle v^l - v, w \rangle \rightarrow 0$ as $l \rightarrow \infty$ for every $w \in \mathcal{H}^k$. We call v the weak limit of $\{v^l\}_{l=1}^\infty$. A set $\mathcal{Q} \subset \mathcal{H}^k$ is sequentially weakly closed if it contains the weak limit of every weakly convergent sequence $\{v^l\}_{l=1}^\infty \subset \mathcal{Q}$.

Lemma 2.6.

- (i) The admissible set \mathcal{A}_p^k is sequentially weakly closed in \mathcal{H}^k .
(ii) Let b be a continuous tensor field of type $(0, s)$ on the manifold N ; and, for every $w \in \mathcal{A}_p^k$, let ψ^w denote the restriction to M of the time-one map of the evolution equation $dq/dt = w(q, t)$. If a sequence $\{v^l\}_{l=1}^\infty \subset \mathcal{A}_p^k$ converges weakly to $v \in \mathcal{A}_p^k$ in \mathcal{H}^k , then

$$\lim_{l \rightarrow \infty} \|(\psi^{v^l})^* b - (\psi^v)^* b\|_{(p_0)} = 0$$

for every $p_0 \in M$.

Proof. (i) Let $\{v^l\}_{l=1}^\infty \subset \mathcal{A}_p^k$ and suppose that v^l converges weakly to some $v \in \mathcal{H}^k$ as $l \rightarrow \infty$. We will show that $v \in \mathcal{A}_p^k$.

By Lemma B.3, $\eta^{v^l}(t; t_0, x) \rightarrow \eta^v(t; t_0, x)$ as $l \rightarrow \infty$ (in the Euclidean norm) for all $t, t_0 \in [0, 1]$ and $x \in \Omega$. Thus, the time-one maps generated by v^l and their inverses converge pointwise: $\phi^{v^l}(x) \rightarrow \phi^v(x)$ and $(\phi^{v^l})^{-1}(x) \rightarrow (\phi^v)^{-1}(x)$ as $l \rightarrow \infty$ for all $x \in \Omega$. Because the manifolds M and N are compact, $\phi^v(M) \subset N$ and $(\phi^v)^{-1}(N) \subset M$. In view of these inclusions, the C^r diffeomorphism ϕ^v of Ω restricted to M is a diffeomorphism between M and N ; in symbols, $\psi^v = \phi^v|_M \in \text{Diff}^r(M, N)$.

By passing to the limit as $l \rightarrow \infty$ in the inequality $\|v\|_{\mathcal{H}^k}^2 \leq \langle v - v^l, v \rangle_{\mathcal{H}^k} + P\|v\|_{\mathcal{H}^k}$, it follows that $\|v\|_{\mathcal{H}^k} \leq P$. Therefore $v \in \mathcal{A}_p^k$, as required.

(ii) For simplicity, let us assume that $s = 2$. Let (U, ξ) be a chart on M at p_0 . It suffices to show that

$$B^l := (\phi^{v^l})^* b(X, Y)(p_0) - (\phi^v)^* b(X, Y)(p_0) \rightarrow 0 \quad (11)$$

as $l \rightarrow \infty$ for all smooth vector fields X, Y on U .

Using the notation

$$\begin{aligned} q^l &= \phi^{v^l}(p_0), \\ q &= \phi^v(p_0), \\ Z_y^l &= D\phi^{v^l} X \circ (\phi^{v^l})^{-1}(y), \\ Z_y &= D\phi^v X \circ (\phi^v)^{-1}(y), \\ Q_y^l &= D\phi^{v^l} Y \circ (\phi^{v^l})^{-1}(y), \quad \text{and} \\ Q_y &= D\phi^v Y \circ (\phi^v)^{-1}(y) \end{aligned}$$

for all $y \in N$, the quantity B^l in expression (11) is recast in the form

$$\begin{aligned} B^l &= b(q^l)(Z_{q^l}^l, Q_{q^l}^l) - b(q)(Z_q, Q_q) \\ &= b(q^l)(Z_{q^l}^l, Q_{q^l}^l) - b(q^l)(Z_{q^l}, Q_{q^l}^l) \end{aligned} \quad (12)$$

$$+ b(q^l)(Z_{q^l}, Q_{q^l}^l) - b(q^l)(Z_{q^l}, Q_{q^l}) \quad (13)$$

$$+ b(q^l)(Z_{q^l}, Q_{q^l}) - b(q)(Z_q, Q_q). \quad (14)$$

Using Definition 2.3 and noting that the Riemannian metric g_N is inherited from \mathbb{R}^{n+1} , we estimate difference (12) as follows:

$$b(q^l)(Z_{q^l}^l, Q_{q^l}^l) - b(q^l)(Z_{q^l}, Q_{q^l}^l) \leq \|b(q^l)\|_{g_N(q^l)} |Z_{q^l}^l - Z_{q^l}|_{\mathbb{R}^{n+1}} |Q_{q^l}^l|_{\mathbb{R}^{n+1}}.$$

By Lemma B.3, $|Z_{q^l}^l - Z_{q^l}|_{\mathbb{R}^{n+1}} \rightarrow 0$ as $l \rightarrow \infty$, and $|Q_{q^l}^l|_{\mathbb{R}^{n+1}}$ is uniformly bounded in $l \in \mathbb{N}$. By Lemma 2.5, there exists a constant $C > 0$ such that $\|b(q^l)\|_{g_N(q^l)} \leq C$ for all $l \in \mathbb{N}$. Therefore, difference (12) converges to zero as $l \rightarrow \infty$. Similarly, it can be shown that difference (13) converges to zero as $l \rightarrow \infty$. Difference (14) converges to zero as $l \rightarrow \infty$ because $z \mapsto b(z)(Z_z, Q_z)$ is a continuous function on U . Hence, $B^l \rightarrow 0$ as $l \rightarrow \infty$. \square

We say that a functional $I : \mathcal{H}^k \rightarrow \mathbb{R}$ is weakly continuous on \mathcal{H}^k if $I(v^l) \rightarrow I(v)$ whenever the sequence $\{v^l\}_{l=1}^\infty \subset \mathcal{H}^k$ converges weakly to v in \mathcal{H}^k .

Recall that the inequality $k \geq (n+1)/2 + r + 1$ guarantees the embedding of the Sobolev space $W_0^{k,2}(\Omega, \mathbb{R}^{n+1})$ into $C^r(\bar{\Omega}, \mathbb{R}^{n+1})$, where $r \geq 1$.

Lemma 2.7. Assume that the constant $P > 0$ is large enough so that the set \mathcal{A}_p^k is not empty. Let the functionals J, I_1, I_2 be defined as in Definition 2.2.

- (i) If $k \geq (n+1)/2 + 3$, then the functionals $J : \mathcal{A}_p^k \rightarrow \mathbb{R}_+$ and $I_1 : \mathcal{A}_p^k \rightarrow \mathbb{R}_+$ are weakly continuous.
(ii) If $k \geq (n+1)/2 + 4$, then the functional $I_2 : \mathcal{A}_p^k \rightarrow \mathbb{R}_+$ is weakly continuous.

Proof. Let $\{v^l\}_{l=1}^\infty \subset \mathcal{A}_p^k$ and suppose that v^l converges weakly to some $v \in \mathcal{H}^k$ as $l \rightarrow \infty$ (in symbols $v^l \rightharpoonup v \in \mathcal{H}^k$). By Lemma 2.6, $v \in \mathcal{A}_p^k$ and $J(v)$, $I_1(v)$, and $I_2(v)$ are well defined.

(i) We will show that $\lim_{l \rightarrow \infty} J(v^l) = J(v)$.

Let $G := g_M^* \otimes g_M^*$. For tensor fields $a, b \in T^0_2(M)$ and every $p \in M$, we have the equality

$$\|a\|^2(p) - \|b\|^2(p) = |G(a+b, a-b)(p)| \leq \|a+b\|(p) \|a-b\|(p).$$

By applying the Cauchy–Schwarz inequality, we obtain the inequality

$$\begin{aligned} |J(v^l) - J(v)| &\leq \int_M \|(\psi^{v^l})^* \tau_N + (\psi^v)^* \tau_N - 2\tau_M\| \|(\psi^{v^l})^* \tau_N - (\psi^v)^* \tau_N\| \omega_M \\ &\leq \left(\int_M \|(\psi^{v^l})^* \tau_N + (\psi^v)^* \tau_N - 2\tau_M\|^2 \omega_M \right)^{1/2} \left(\int_M \|(\psi^{v^l})^* \tau_N - (\psi^v)^* \tau_N\|^2 \omega_M \right)^{1/2}. \end{aligned} \quad (15)$$

By Lemma 2.6,

$$\lim_{l \rightarrow \infty} \|(\psi^{v^l})^* \tau_N - (\psi^v)^* \tau_N\|^2(p) = 0 \quad (16)$$

for all $p \in M$.

Let $K > 0$ be the constant in display (42) of Lemma B.3. By Lemma 2.5 and because the manifold N is compact, there exists a constant $C > 0$ such that $\|\tau_N(z)\|_{g_N(z)} \leq C$ for all $z \in N$. Using the equivalence of norms (10), we estimate

$$\begin{aligned} \|(\psi^{v^l})^* \tau_N\|(p) &\leq n^{s/2} \|(\psi^{v^l})^* \tau_N(p)\|_{g_M(p)} \\ &\leq n^{s/2} \|\tau_N(\psi^{v^l}(p))\|_{g_N(\psi^{v^l}(p))} |D\psi^{v^l}(p)|^s \\ &\leq n^{s/2} CK^s. \end{aligned} \quad (17)$$

Using inequalities (15) and (17), limit (16), and the Dominated Convergence Theorem, we conclude that $J(v^l) \rightarrow J(v)$ as $l \rightarrow \infty$.

Let us show that the functional I_1 is weakly continuous. By an estimate analogous to (15), it suffices to prove the following statements.

- (I) If $p \in M$ and $t \in [0, 1]$, then

$$\lim_{l \rightarrow \infty} \|F^{v^l}(\cdot, t)^* g_t^{v^l} - F^v(\cdot, t)^* g_t^v\|^2(p) = 0.$$

- (II) There exists $S_1 > 0$ such that

$$\|F^{v^l}(\cdot, t)^* g_t^{v^l}\|^2(p) \leq S_1$$

for all $p \in M$, $t \in [0, 1]$, and $l \in \mathbb{N}$.

Because all the Riemannian metrics are inherited from \mathbb{R}^{n+1} , whose standard inner product is denoted by $\langle \cdot, \cdot \rangle$, we have

$$F^{v^l}(\cdot, t)^* g_t^{v^l}(p)(X, Y) - F^v(\cdot, t)^* g_t^v(p)(X, Y) = \langle D_x F^{v^l}(p, t)X, D_x F^{v^l}(p, t)Y \rangle - \langle D_x F^v(p, t)X, D_x F^v(p, t)Y \rangle$$

for all $p \in M$, $X, Y \in T_p M$, and $t \in [0, 1]$, where D_x denotes the derivative with respect to the spatial variable. The right-hand side of this equation converges to zero as $l \rightarrow \infty$ by Lemma B.3. This completes the proof of statement (I).

By the same lemma and inequality (10), for every $p \in M$ we have

$$\begin{aligned} \|F^{v^l}(\cdot, t)^* g_t^{v^l}\|(p) &\leq n \|F^{v^l}(\cdot, t)^* g_t^{v^l}(p)\|_{g_M(p)} \\ &\leq n |D_x F^{v^l}(p, t)|^2 \\ &\leq nK^2. \end{aligned}$$

This inequality implies statement (II).

(ii) We will show the weak continuity of the functional I_2 . By an estimate analogous to (15), it suffices to show two facts:

(III) If $p \in M$ and $t \in [0, 1]$, then

$$\lim_{l \rightarrow \infty} \|F^{v^l}(\cdot, t)^* \Pi_t^{v^l} - F^v(\cdot, t)^* \Pi_t^v\|^2(p) = 0.$$

(IV) There exists $S_2 > 0$ such that

$$\|F^{v^l}(\cdot, t)^* \Pi_t^{v^l}\|^2(p) \leq S_2$$

for all $p \in M$, $t \in [0, 1]$ and $l \in \mathbb{N}$.

We will first prove statement (IV).

Consider a morph F^w generated by a time-dependent vector field $w \in \mathcal{A}_p^k$. By definition of morphs of class $\mathcal{M}^{l,ac}(M, N)$ in Section 1.3, the orientation of each intermediate manifold $M^{w,t}$, where $t \in [0, 1]$, is induced by the C^2 diffeomorphism $F(\cdot, t) : M \rightarrow M^{w,t}$. Let $\mathcal{N}^{w,t}(z)$ denote the unit normal to the manifold $M^{w,t}$ at the point $z \in M^{w,t}$. We assume that for every positively oriented basis $\{X_i\}_{i=1}^n$ of $T_z M^{w,t}$, the set of vectors $\{X_1, \dots, X_n, \mathcal{N}^{w,t}(z)\}$ is positively oriented in \mathbb{R}^{n+1} .

For $p \in M$, let (U, ξ) be a chart at p , choose two smooth vector fields X and Y on U , and let $\gamma : [0, 1] \rightarrow U$ be a C^1 curve at p such that $\dot{\gamma}(0) = X_p$. It is evident that the inner product

$$\langle \mathcal{N}^{w,t}(F^w(\gamma(s), t)), D_x F^w(\gamma(s), t) Y_{\gamma(s)} \rangle = 0 \quad (18)$$

for every $t, s \in [0, 1]$. Let us recall that for every $t \in [0, 1]$ the function $x \mapsto F^{w,t}(x, t)$ is defined for all $x \in \Omega$ and denote its second derivative at $x \in \Omega$ by $D_x^2 F(x, t)$. By differentiating expression (18) with respect to s at $s = 0$, we obtain the equality

$$\begin{aligned} F^w(\cdot, t)^* \Pi_t^w(p)(X_p, Y_p) &= \langle \bar{\nabla}_{D_x F^w(p,t)X_p} \mathcal{N}^{w,t}(F^w(p, t)), D_x F^w(p, t) Y_p \rangle \\ &= -\langle \mathcal{N}^{w,t}(F^w(p, t)), D_x^2 F^w(p, t)[X_p, Y_p] \rangle, \end{aligned} \quad (19)$$

where $\bar{\nabla}$ denotes the standard Riemannian connection on \mathbb{R}^{n+1} (see [12]).

For every $p \in M$, let $W_p, Q_p \in T_p M$ be unit length vectors such that

$$\|F^{v^l}(\cdot, t)^* \Pi_t^{v^l}(p)\|_{g_M(p)} = |F^{v^l}(\cdot, t)^* \Pi_t^{v^l}(p)(W_p, Q_p)|.$$

Using inequality (10) and Eq. (19), we have the estimates

$$\begin{aligned} \|F^{v^l}(\cdot, t)^* \Pi_t^{v^l}\|(p) &\leq n \|F^{v^l}(\cdot, t)^* \Pi_t^{v^l}(p)\|_{g_M(p)} \\ &= n |F^{v^l}(\cdot, t)^* \Pi_t^{v^l}(p)(W_p, Q_p)| \\ &= n |\langle \mathcal{N}^{v^l,t}(F^{v^l}(p, t)), D_x^2 F^{v^l}(p, t)[W_p, Q_p] \rangle| \\ &\leq nK. \end{aligned}$$

This completes the proof of statement (IV).

By Lemma B.3, if $\alpha \in [0, 1, 2]$, then the derivative $D_x^\alpha F^{v^l}(p, t)$ converges to $D_x^\alpha F^v(p, t)$ as $l \rightarrow \infty$ in the Euclidean norm for every $p \in M$ and $t \in [0, 1]$. Taking into account Eq. (19), we see that statement (III) follows from the convergence

$$\mathcal{N}^{v^l,t}(F^{v^l}(p, t)) \rightarrow \mathcal{N}^{v,t}(F^v(p, t)) \quad (20)$$

as $l \rightarrow \infty$ in \mathbb{R}^{n+1} for every $p \in M$ and $t \in [0, 1]$. \square

Theorem 2.8. Assume that the constant $P > 0$ is large enough so that the set \mathcal{A}_p^k is not empty.

- (i) If $k \geq (n+1)/2 + 3$, then there exists a minimizer of the bending distortion energy functional E in the admissible set \mathcal{A}_p^k .
- (ii) If $k \geq (n+1)/2 + 4$, then there exists a minimizer of the morphing distortion energy functional \mathcal{E} in the admissible set \mathcal{A}_p^k .

Proof. Let $\{v^l\}_{l=1}^\infty \subset \mathcal{A}_p^k$ be a minimizing sequence of E , that is

$$\lim_{l \rightarrow \infty} E(v^l) = \inf_{w \in \mathcal{A}_p^k} E(w) \geq 0.$$

By Lemma 2.6, the set \mathcal{A}_p^k is sequentially weakly closed and bounded. Therefore, there exists a weakly convergent subsequence $\{v^{l_k}\}_{k=1}^\infty$ with the weak limit $v \in \mathcal{A}_p^k$. The functional E is weakly continuous by Lemma 2.7. Therefore, $E(v) = \inf_{w \in \mathcal{A}_p^k} E(w)$ and v is a minimizer of E .

The existence of minimizers for the functional \mathcal{E} is proved in the same fashion. \square

Remark 2.9. Theorem 2.8 implies the existence of minimizers of the functional Λ defined in display (2) in the admissible set

$$\mathcal{B}_p^k := \{h \in \text{Diff}^2(M, N) : h = \psi^v \text{ for some } v \in \mathcal{A}_p^k\}.$$

The set \mathcal{B}_p^k , among other maps, contains smooth diffeomorphisms $f : M \rightarrow N \subset \mathbb{R}^{n+1}$ that are homotopic to the inclusion map $i : M \rightarrow \mathbb{R}^{n+1}$ and generate time-dependent vector fields in \mathcal{A}_p^k .

To minimize the distortion energy of diffeomorphisms from other isotopy classes, we consider the family of maps $\{\psi^v \circ \phi \in \text{Diff}^r(M, N) : v \in \mathcal{A}_p^k\}$, where ϕ is a fixed diffeomorphism of M . Similarly, given a smooth isotopy $G : [0, 1] \times M \rightarrow M$, we consider the family of morphs $\{F_G^v \in \mathcal{M}^{r, \text{ac}}(M, N) : v \in \mathcal{A}_p^k\}$, where $F_G^v(p, t) = F^v(G(p, t), t)$ for all $(p, t) \in M \times [0, 1]$, as candidates for minimal distortion morphs. The most interesting example of this generalization is, perhaps, the case where $G(p, t) = \phi(p)$ for some fixed diffeomorphism $\phi : M \rightarrow N$, so that the admissible isotopies are from the class of morphs $F^v(\phi(p), t)$ generated by time-dependent vector fields in \mathcal{A}_p^k , where $p \in M$ and $t \in [0, 1]$.

The latter idea leads to the definition of the functionals

$$E_\phi(v) = E_\phi(v; B_1, B_2) = B_1 \int_M \|(\psi^v \circ \phi)^* g_N - g_M\|^2 \omega_M + B_2 \int_M \|(\psi^v \circ \phi)^* \Pi_N - \Pi_M\|^2 \omega_M$$

and

$$\mathcal{E}_G(v) = \mathcal{E}_G(v; B_1, B_2) = B_1 \int_0^1 \int_M \|F_G^v(\cdot, t)^* g_t^v - g_M\|^2 \omega_M dt + B_2 \int_0^1 \int_M \|F_G^v(\cdot, t)^* \Pi_t^v - \Pi_M\|^2 \omega_M dt,$$

where B_1 and B_2 are nonnegative real numbers (cf. Definition 1.3), $\phi \in \text{Diff}(M)$ and $G : M \times [0, 1] \rightarrow M$ is an isotopy.

Theorem 2.8 can be easily generalized to show that for $P > 0$ and $k \in \mathbb{N}$ sufficiently large, both functionals E_ϕ and \mathcal{E}_G have minimizers in \mathcal{A}_p^k for every diffeomorphism $\phi : M \rightarrow M$ and isotopy $G : M \times [0, 1] \rightarrow M$.

Theorem 2.10. Assume that the constant $P > 0$ is large enough so that the set \mathcal{A}_p^k is not empty. Let $\phi \in \text{Diff}(M)$ and let $G : M \times [0, 1] \rightarrow M$ be an isotopy.

- (i) If $k \geq (n+1)/2 + 3$, then there exists a minimizer of the bending distortion energy functional E_ϕ in the admissible set \mathcal{A}_p^k .
- (ii) If $k \geq (n+1)/2 + 4$, then there exists a minimizer of the morphing distortion energy functional \mathcal{E}_G in the admissible set \mathcal{A}_p^k .
- (iii) If $k \geq (n+1)/2 + 3$, then both functionals Φ and Λ defined in displays (1) and (2) respectively have minimizers in the admissible set

$$\mathcal{B}_{p,\phi}^k := \{h \in \text{Diff}^2(M, N) : h = \psi^v \circ \phi \text{ for some } v \in \mathcal{A}_p^k\}.$$

The latter theorem is an easy generalization of Theorem 2.8. More precisely, let $\{b^l\}_{l=1}^\infty$ be a sequence of tensor fields in $\mathcal{T}_s^0(M)$ such that $\lim_{l \rightarrow \infty} \|b^l\|(p) = 0$ and $\|b^l(p)\| \leq K$ for all $p \in M$ and $l \in \mathbb{N}$, where K is a positive constant and let $\phi \in \text{Diff}(M)$. Then $\lim_{l \rightarrow \infty} \|\phi^* b^l\|(p) = 0$ and there exists a constant $K_1 > 0$ such that $\|\phi^* b^l(p)\| \leq K_1$ for all $p \in M$ and $l \in \mathbb{N}$. Using the above observation, Lemma 2.7 is easily generalized to the case where ψ^v and F^v are replaced with $\psi^v \circ \phi$ and F_G^v respectively, and the proof of Theorem 2.8 remains the same.

In Theorem 2.10, the statement (iii), which is equivalent to Theorem 1.5, follows from the statement (i).

The problem of finding minimizers of the functional \mathcal{E} does not have a straightforward solution. Even after we restrict our attention to the family of morphs whose intermediate states are concentric circles, finding a minimal distortion morph requires delicate analysis. An example of a minimal distortion morph between two circles and construction of minimizers of the functionals E and \mathcal{E} between other compact isotopic manifolds are subjects of a future paper.

3. Minimal deformation bending of two-dimensional spheres

In this section we minimize the deformation energy functional Φ defined in display (1) under the assumptions $M = \mathbb{S}^2$ and $N = h_R(\mathbb{S}^2) = R\mathbb{S}^2$ for some $R > 0$, where \mathbb{S}^2 is the unit 2-dimensional sphere in \mathbb{R}^3 , and h_R is the radial map defined by $h_R(y) = Ry$ for all $y \in \mathbb{R}^3$. As usual, the manifolds M and N are equipped with Riemannian metrics g_M and g_N respectively induced by the Euclidean metric $dx_1^2 + dx_2^2 + dx_3^2$ of \mathbb{R}^3 . The manifolds M and N are Riemann surfaces (see [16, 14]). We parametrize the spheres \mathbb{S}^2 and $R\mathbb{S}^2$ on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ using stereographic projections. For $(y_1, y_2, y_3) \in \mathbb{S}^2$, the stereographic projection is given by the expression $\pi(y_1, y_2, y_3) = \frac{y_1 + iy_2}{1 - y_3}$. We will show that maps of the form $h = f \circ h_R|_M$, where f is an isometry on N , minimize the functional Φ defined by Eq. (1) in the class of all holomorphic diffeomorphisms from M to N .

We note that a holomorphic minimizer $h : M \rightarrow N$ is a critical point of the functional Φ on its natural domain $\text{Diff}(M, N)$ (see Appendix A and Corollary A.4).

The parametrization $\phi : \hat{\mathbb{C}} \rightarrow \mathbb{S}^2$ is given by

$$\phi(u + iv) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right)^T, \quad (21)$$

and the parametrization $\phi_R : \hat{\mathbb{C}} \rightarrow R\mathbb{S}^2$ of $R\mathbb{S}^2$ is given by $\phi_R(u + iv) = R\phi(u + iv)$.

In these coordinates, the Riemannian metrics g_M and g_N are defined by

$$g_M(z, \bar{z}) = \frac{4}{(1 + |z|^2)^2} dz d\bar{z} \quad (22)$$

and

$$g_N(z, \bar{z}) = \frac{4R^2}{(1 + |z|^2)^2} dz d\bar{z}. \quad (23)$$

Let $h \in \text{Diff}(M, N)$ be a holomorphic map. The local representation $(\phi_R)^{-1} \circ h \circ \phi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of h , which (by an abuse of notation) we shall denote by the same letter, is a holomorphic diffeomorphism of the extended complex plane onto itself. We conclude that $h(z)$ has the form $h(z) = M(z)$, where $M(z) = \frac{az+b}{cz+d}$ is a Möbius transformation and $a, b, c, d \in \mathbb{C}$ are such that $ad - bc \neq 0$. For such an h , it is easy to derive the formula

$$h^* g_N(z, \bar{z}) = \frac{4R^2 |bc - ad|^2}{(|az + b|^2 + |cz + d|^2)^2} dz d\bar{z}. \quad (24)$$

Hence, the problem of minimization of the deformation energy functional Φ defined in display (1) over all holomorphic diffeomorphisms from \mathbb{S}^2 to $R\mathbb{S}^2$ reduces to the problem of minimization of the function

$$\Psi(a, b, c, d) = \int_{\mathbb{R}^2} \left(\frac{R^2 |bc - ad|^2}{(|az + b|^2 + |cz + d|^2)^2} - \frac{1}{(1 + |z|^2)^2} \right)^2 (1 + |z|^2)^2 du dv, \quad (25)$$

where $z = u + iv$, over the group $\text{Aut}(\hat{\mathbb{C}}) = \text{PGL}(2, \mathbb{C})$. Recall that the elements of the projective general linear group $\text{PGL}(2, \mathbb{C})$ are the equivalence classes $[a, b, c, d]$, where $ad - bc \neq 0$ and $(a', b', c', d') \in [a, b, c, d]$ if $(a', b', c', d') = \lambda(a, b, c, d)$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Recall that the group of all isometries of the Riemann sphere is the projective unitary group $\text{PU}(2, \mathbb{C})$; that is, every isometry f of (\mathbb{S}^2, g_M) has the local representation (via stereographic projection)

$$f(z) = \frac{az - \bar{c}}{cz + \bar{a}},$$

where $a, c \in \mathbb{C}$ are such that $|a|^2 + |c|^2 = 1$.

The functional Φ is invariant with respect to left compositions with isometries; that is, $\Phi(f \circ h) = \Phi(h)$ for every isometry $f \in \text{Diff}(N)$ and $h \in \text{Diff}(M, N)$. Therefore, the reduced function Ψ is well defined on the quotient of $\text{PGL}(2, \mathbb{C})$ by $\text{PU}(2, \mathbb{C})$, which is the set of all equivalence classes

$$\left[\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right] = \left\{ \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \in \text{PU}(2, \mathbb{C}) \right\}. \quad (26)$$

We note that the equivalence class

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

consists of all the isometries of the unit sphere (\mathbb{S}^2, g_M) .

Proof of Theorem 1.2. Statement (ii) of Theorem 1.2 follows immediately from Hurwitz's automorphisms theorem: *The group of automorphisms of a compact Riemann surface of genus greater than one is finite* (see [16]).

Statement (i) of Theorem 1.2 is equivalent to the following result.

Theorem 3.1. *The equivalence class of the isometries of (\mathbb{S}^2, g_M) is the unique minimizer of the function Ψ defined on the homogeneous space $\text{PGL}(2, \mathbb{C}) / \text{PU}(2, \mathbb{C})$; that is,*

$$\Psi \left(\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \right) \leq \Psi \left(\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right] \right) \quad (27)$$

for all

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}(2, \mathbb{C}) / \mathrm{PU}(2, \mathbb{C}).$$

Proof. The function Ψ is well defined on the homogeneous space $\mathrm{PGL}(2)/\mathrm{PU}(2)$. Thus, all values of Ψ are obtained by choosing its domain to consist of one representative from each equivalence class.

We claim that each equivalence class

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

has a representative of the form

$$\begin{pmatrix} 1 & 0 \\ z & r \end{pmatrix}$$

for some $z \in \mathbb{C}$ and $r \in \mathbb{R}_+$.

To prove the claim, note that (without loss of generality) we may assume the determinant of the given representative is unity; that is, $\alpha\delta - \beta\gamma = 1$. We wish to prove the existence of $a, c \in \mathbb{C}$ so that

$$\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & r \end{pmatrix} \quad (28)$$

for some $z \in \mathbb{C}$ and $r \in \mathbb{R}_+$. In other words, it suffices to solve the system of linear equations

$$\begin{cases} a\alpha - \bar{c}\gamma = 1, \\ a\beta - \bar{c}\delta = 0. \end{cases} \quad (29)$$

In view of the equation $\alpha\delta - \beta\gamma = 1$, it follows that $a = \delta$ and $c = \bar{\beta}$. By substitution of a and c into Eq. (28), we find that $z = \bar{\beta}\alpha + \bar{\delta}\gamma$ and $r = |\beta|^2 + |\delta|^2$. This proves the claim.

By the claim, it suffices to consider the value of Ψ only at points of the form $(1, 0, qe^{i\psi}, r)$, where $q \in \mathbb{R}$, $r \in \mathbb{R}_+$, and $\psi \in [0, 2\pi)$. Thus, the theorem is an immediate consequence of the following proposition.

The function $\bar{\Psi} : \mathbb{R} \times [0, 2\pi] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ given by

$$\bar{\Psi}(q, \psi, r) = \Psi(1, 0, qe^{i\psi}, r) \quad (30)$$

attains its global minimum on the set of points $(0, \psi, 1)$.

To prove this result, let us first calculate the integral that represents the function $\bar{\Psi}$.

After passing to polar coordinates ($u = \rho \cos \phi$ and $v = \rho \sin \phi$), we represent $\bar{\Psi}$ in the form

$$\bar{\Psi}(q, \psi, r) = \int_0^\infty \int_0^{2\pi} \left[\frac{R^2 r^2}{(\xi + \eta \cos(\phi + \psi))^2} - \frac{1}{(1 + \rho^2)^2} \right]^2 (1 + \rho^2)^2 \rho d\phi d\rho,$$

where $\xi = \rho^2 + \rho^2 q^2 + r^2$ and $\eta = 2\rho qr$. Since the integrand is periodic with respect to ϕ and we are integrating over one period, $\bar{\Psi}(q, \psi, r)$ does not depend on ψ ; that is,

$$\bar{\Psi}(q, \psi, r) = \int_0^\infty \int_0^{2\pi} \left[\frac{R^2 r^2}{(\xi + \eta \cos \phi)^2} - \frac{1}{(1 + \rho^2)^2} \right]^2 (1 + \rho^2)^2 \rho d\phi d\rho.$$

The inner integral of the equivalent iterated integral is

$$\begin{aligned} K(\rho) &:= \int_0^{2\pi} \left[\frac{R^2 r^2}{(\xi + \eta \cos(\phi))^2} - \frac{1}{(1 + \rho^2)^2} \right]^2 (1 + \rho^2)^2 \rho d\phi \\ &= \left[R^4 r^4 \int_0^{2\pi} \frac{1}{(\xi + \eta \cos(\phi))^4} d\phi - 2R^2 r^2 \frac{1}{(1 + \rho^2)^2} \int_0^{2\pi} \frac{1}{(\xi + \eta \cos(\phi))^2} d\phi + 2\pi \frac{1}{(1 + \rho^2)^4} \right] \rho (1 + \rho^2)^2. \end{aligned} \quad (31)$$

Taking into account the inequalities $\xi > |\eta|$ and $\eta > 0$, the integrals in the previous expression are elementary; their values are given by

$$\int_0^{2\pi} \frac{1}{(\xi + \eta \cos(\phi))^2} d\phi = \frac{2\pi\xi}{(\xi^2 - \eta^2)^{3/2}}$$

and

$$\int_0^{2\pi} \frac{1}{(\xi + \eta \cos(\phi))^4} d\phi = \frac{\pi\xi(2\xi^2 + 3\eta^2)}{(\xi^2 - \eta^2)^{7/2}}.$$

By substitution into Eq. (31), we find that

$$\begin{aligned} \bar{\Psi}(q, \psi, r) &= \int_0^\infty K(\rho) d\rho \\ &= \pi - 2\pi R^2 + \frac{\pi R^4}{3r^2} (1 + q^2 + (r-1)r)(1 + q^2 + r + r^2). \end{aligned} \quad (32)$$

The minimum of the function $F(q, r) = \bar{\Psi}(q, \psi, r)$ on $\mathbb{R} \times \mathbb{R}_+$, is easily determined. Indeed, $(0, 1)$ is the only critical point of F . Also, the Hessian of F is

$$D^2F(q, r) = \begin{pmatrix} \frac{4\pi R^4(1+3q^2+r^2)}{3r^2} & -\frac{8\pi R^4q(1+q^2)}{3r^3} \\ -\frac{8\pi R^4q(1+q^2)}{3r^3} & \frac{2\pi R^4(3+6q^2+3q^4+r^4)}{3r^4} \end{pmatrix}.$$

We note that $\partial^2 F(q, r)/\partial q^2$ and the determinant of the Hessian

$$\det(D^2F(q, r)) = \frac{8\pi^2 R^8}{9r^6} (3 + q^6 + 3r^2 + r^4 + r^6 + q^4(5 + 3r^2) + q^2(7 + 6r^2 + 3r^4))$$

are both positive by inspection. By Sylvester's criterion, the Hessian is positive definite over the entire domain of F ; therefore, F is convex. It follows that $(0, 1)$ is the unique global minimizer of F . The minimum of F is

$$F(0, 1) = \pi(R^2 - 1)^2.$$

Clearly, points of the form $(0, \psi, 1)$ are the global minima of $\bar{\Psi}$ on its domain $\mathbb{R} \times [0, 2\pi] \times \mathbb{R}_+$. \square

4. Discussion

For diffeomorphic hypersurfaces M and N in a Euclidean space, we have defined functionals that measure how well a diffeomorphism $\psi : M \rightarrow N$ preserves the geometry of M and proved that minimizers of these functionals exist. Since our functionals involve comparisons of the first and the second fundamental forms on M with the pull-backs of the corresponding fundamental forms from N to M by ψ , they measure the extent to which ψ changes the size of vectors carried from M to N and the extent to which it preserves the amount of bending of the unit normal vector field on M . In addition, to maintain flexibility for applications (where some particular combination of the measurements given by the first and the second fundamental forms is desired) we allow the measurement of distortion length and distortion bending to be weighted.

Even in the case where M and N are one-dimensional, there are examples of such manifolds where the distortion energy functional for the first fundamental form has no minimizer in $\text{Diff}(M, N)$ (see [3]). This fact and the complexity of the corresponding Euler–Lagrange equation (derived in Appendix A) evidence the difficulty of the general problem of minimization of distortion energy functionals over $\text{Diff}(M, N)$.

One of the main ideas used successfully here is to restrict, via an appropriate linearization, the space of diffeomorphisms on which the distortion energy functionals are minimized. More precisely, instead of working directly with maps in $\text{Diff}(M, N)$, we consider time-dependent vector fields on the ambient Euclidean space that generate the desired diffeomorphisms. Each such vector field determines an evolution family that can be applied to M , for values of the evolution parameter in the interval $[0, 1]$, to produce a morph carrying M to some diffeomorphic end-hypersurface. We restrict our attention to the class of vector fields for which this end-hypersurface is N . The advantage is clear: The elements of $\text{Diff}(M, N)$ are replaced by elements of a vector space. Using this approach, which already appears in the literature on image deformation (see [7,9,11,18]), we are able to prove the existence of minimizers for the distortion morphing functional \mathcal{E} and the distortion bending functional E . The functional \mathcal{E} is an appropriate choice to measure distortion in case we wish to include the deviation of the geometry of the intermediate surfaces from the original surface M . On the other hand, the distortion functional E is appropriate if we wish to ignore the distortions represented by the intermediate surfaces and only consider the distortion caused by mapping M to N .

We have determined minimizers of our distortion energy functionals for Riemann surfaces. While this special case has independent interest, it also serves to test conjectures.

The existence of minimizers provides a hunting license for seeking approximations of these minimal distortion diffeomorphisms and morphs via numerical methods. The known minimizers for special cases can be used to test numerical algorithms.

Our admissible set \mathcal{A}_P^k of time-dependent vector fields is the closed ball of radius $P > 0$ in the Hilbert space \mathcal{H}^k of vector fields intersected with the set \mathcal{B} of all vector fields that carry M to N . The boundedness of \mathcal{A}_P^k in \mathcal{H}^k is an essential ingredient in our proof of the existence of distortion energy minimizers. Thus, there is a natural (open) question: Is there a minimizer in the interior of \mathcal{A}_P^k (in the relative topology of \mathcal{B}) or are all minimizers on the boundary of this set?

We mention some evidence that suggests there are energy minimizers of the functional E in the interior of \mathcal{A}_P^k for $P > 0$ sufficiently large. We expect the same result for \mathcal{E} , but this case seems to be much more subtle.

In [3] it was shown that the radial map is a minimal distortion map in $\text{Diff}(\mathbb{S}^1, \mathbb{S}_R^1)$ in case $R > 1$. It is easy to construct a time-dependent vector field whose time-one map is the radial map. Hence, at least for this special case, interior minimizers exist. Indeed, for P larger than the Hilbert-norm of our time-dependent vector field that produces the radial map, this vector field is in the interior of the admissible set \mathcal{A}_P^k .

To prove the existence of an interior minimizer in general, it suffices to show that there is a number $P_0 > 0$ such that for every time-dependent vector field $v \in \mathcal{H}^k$ that carries M to N such that its Hilbert-norm $\|v\|_{\mathcal{H}^k} \geq P_0$, there exists a vector field v' whose Hilbert-norm is less than P_0 and whose energy (as measured by E or \mathcal{E}) does not exceed the corresponding energy of v .

Suppose that for some large P we have a minimizer of the functional E , a time-dependent vector field v whose Hilbert-norm is P ; that is, v lies on the boundary of the admissible set. The time-one map of v determines the value of E . We seek a new vector field v' with smaller Hilbert-norm and the same time-one map as v ; it would be our desired interior minimizer. A large Hilbert-norm for v suggests that its integral curves have large lengths or largeness in some measure of bending that takes into account the space derivatives of v that are used to compute the Sobolev norm. By shortening and straightening the integral curves of v in some subset of the space $\Omega \times [0, 1]$, where our time-dependent vector fields are defined, we can construct a new time-dependent vector field v' whose Hilbert-norm is strictly smaller than the norm of v and whose time-one map is the same as the time-one map of v . The problem of existence of minimizers of the functionals E and \mathcal{E} in the interior of the admissible set \mathcal{A}_P^k , where $P > 0$ is large enough, is a subject of future research.

We have proved the existence of minimizers of the functionals E and \mathcal{E} in the admissible set \mathcal{A}_P^k for $P > 0$ and $k \in \mathbb{N}$ large enough. Because the values of the functionals E and \mathcal{E} on every given time-dependent vector field $v \in \mathcal{A}_P^k$ depend only on the values of v along the “path” $F^v(M, [0, 1])$ of the morph F^v generated by v , the minimizers of E and \mathcal{E} are not unique. It is reasonable to ask for a minimizer of the cost functional E or \mathcal{E} with minimal \mathcal{H}^k -norm. Such a minimizer, if it exists, is a solution of an additional optimization problem $\|v\|_{\mathcal{H}^k} \rightarrow \min$ subject to the constraint $v \in \mathcal{Q}$, where \mathcal{Q} is the set of all minimizers of the functional E (or \mathcal{E}).

We observe that if we redefine the energy functionals E and \mathcal{E} by setting the new functional equal to the old functional plus the squared \mathcal{H}^k -norm of the argument, the new functionals have minimizers in the admissible set $\{v \in \mathcal{H}^k : \psi^v(M) = N\}$. The proof follows immediately from the coercivity of the new functionals and their weak lower semicontinuity, which is proved in Section 2 of this paper (it is a standard fact that the functional $v \mapsto \|v\|_{\mathcal{H}^k}$ is weakly lower semicontinuous on the Hilbert space \mathcal{H}^k).

Our choice of admissible time-dependent vector fields, which are defined on $\Omega \times [0, 1]$, for Ω a ball in \mathbb{R}^{n+1} containing the manifolds M and N , can be replaced by other sets of functions chosen to not contain extraneous information. For example, it might be desirable to consider only the curves leading from M to N and not all the integral curves generated by our time-dependent vector fields defined on $\Omega \times [0, 1]$. One alternative admissible set is a subset of a class of functions we call *development vector fields*. They are defined to be functions $v : M \times [0, 1] \rightarrow T\mathbb{R}^{n+1}$ (in an appropriate function space) that generate morphs $F^v : M \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ via the following construction. Let $\tau_p^q : T_p\mathbb{R}^{n+1} \rightarrow T_q\mathbb{R}^{n+1}$ be the parallel transport from the tangent space at p to the tangent space at q defined by the Euclidean metric (or perhaps some other metric) on \mathbb{R}^{n+1} . For each $p \in M \subset \mathbb{R}^{n+1}$, the solution of the initial value problem

$$\dot{c}_p = \tau_p^{c_p} v(p, t), \quad c_p(0) = p$$

is a curve in \mathbb{R}^{n+1} starting on M at p . Thus, we have defined a morph $F^v : M \times [0, 1] \rightarrow \mathbb{R}^{n+1}$ given by $F^v(p, t) := c_p(t)$. The new admissible set is all development vector fields that are in some closed ball of radius $P > 0$ of the corresponding Hilbert space such that $F^v(M, 1) = N$.

We also mention the possibility of defining a new norm on our time-dependent or development vector fields by reversing the order of integration:

$$\|v\|^2 := \int_{\mathcal{R}} \int_0^1 \sum_{|\alpha| \leq k} |D_x^\alpha v(x, t)|^2 dt dx,$$

where \mathcal{R} is Ω for time-dependent vector fields and M for development vector fields. This norm may differ from the norm $\|v\|_{\mathcal{H}^k}$ used throughout this paper as functions in the Hilbert space \mathcal{H}^k are not necessarily L^1 in the joint (x, t) variable on $\Omega \times [0, 1]$. Possible advantages of this approach are a better understanding of the relation between this norm and the

shapes of the (integral) curves that are used to define morphs and, by first integrating over the temporal parameter, the Sobolev-norm can be viewed as a norm for functions defined on the compact manifold M . Consideration of this norm led us to the concept of a development vector field. Alternative approaches to the problem of minimal morphing using these ideas are a subject for future research.

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Appendix A. The Euler–Lagrange equation for the deformation energy functional

We will determine the Euler–Lagrange equation for the deformation energy functional $\Phi : \text{Diff}(M, N) \rightarrow \mathbb{R}_+$ defined in display (1). Let $c : (-\varepsilon, \varepsilon) \rightarrow \text{Diff}(M, N)$ be a C^1 curve at $h \in \text{Diff}(M, N)$, which we call a *variation* of h . The equivalence class $[c]_h \in T_h \text{Diff}(M, N)$ can be identified with the smooth vector field $Y \in \mathfrak{X}(N)$ defined by $Y(q) = \frac{d}{dt} c(t) \circ h^{-1}(q)$ for all $q \in N$. We call the vector field $Y \in \mathfrak{X}(N)$ a *variational vector field* of $h \in \text{Diff}(M, N)$. We will compute the first variation $D\Phi(h)Y$ for all directions $Y \in \mathfrak{X}(N)$.

Consider a smooth vector field $X \in \mathfrak{X}(M)$ with flow ϕ_t and a diffeomorphism $h \in \text{Diff}(M, N)$, and suppose that the variation $c(t) = h \circ \phi_t$ induces the variational vector field $Y = h_* X \in \mathfrak{X}(N)$. The diffeomorphism h is a critical point of the functional Φ if and only if

$$\left. \frac{d}{dt} \Phi(h \circ \phi_t) \right|_{t=0} = D\Phi(h)h_* X = 2 \int_M G(h^* g_N - g_M, L_X h^* g_N) \omega_M = 0 \quad (33)$$

for all $X \in \mathfrak{X}(M)$, where $G = g_M^* \otimes g_M^*$.

Let ∇ and $\bar{\nabla}$ be Riemannian connections on M compatible with the Riemannian metrics $\alpha = g_M$ and $\beta = h^* g_N$ respectively, and denote the corresponding Christoffel symbols of ∇ and $\bar{\nabla}$ by Γ_{jk}^i and $\bar{\Gamma}_{jk}^i$.

Let Y be a smooth vector field on M expressed in components by $Y = Y^k \frac{\partial}{\partial x^k}$. The components of the Lie derivative of the Riemannian metric β in the direction of the vector field Y are

$$[L_Y \beta]_{km} = \bar{\nabla}_k Y_m + \bar{\nabla}_m Y_k, \quad (34)$$

where $Y_m = [\beta]_{mj} Y^j$ are the lowered coordinates of Y via the Riemannian metric β (see [1]).

Recall that $T^r_s(M)$ is the set of all continuous tensor fields on M contravariant of order r and covariant of order s , or type (r, s) .

Definition A.1.

- (i) Define the tensor field $B = (\beta - \alpha)^{\#\#} \in T^2_0(M)$. In other words, B equals the strain tensor field $h^* g_N - g_M$ with both indices raised via the Riemannian metric $\alpha = g_M$. Its components are given by $B^{km} = (\beta_{ij} - \alpha_{ij}) \alpha^{ik} \alpha^{jm}$.
- (ii) The bilinear form $A : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is defined by

$$A(X, Y) = \bar{\nabla}_X Y - \nabla_X Y \quad (35)$$

for all $X, Y \in \mathfrak{X}(M)$.

Remark A.2. The bilinear form A can be viewed as a tensor field of type $(1, 2)$ on M with components

$$A^m_{kp} = \bar{\Gamma}^m_{kp} - \Gamma^m_{kp} \quad (36)$$

(see [15, Proposition 7.10]).

Recall that the divergence of a tensor field $\tau \in T^r_s(M)$ is defined to be (see [12])

$$\text{div } \tau = C^r_{s+1}(\nabla \tau), \quad (37)$$

where C^j_i denotes the contraction in lower i and upper j index. The divergence of τ , $\text{div } \tau$, is a tensor of type $(r-1, s)$.

For two tensor fields $\theta \in T^r_2(M)$ and $\tau \in T^2_s(M)$, $\theta : \tau$ denotes the type (r, s) tensor field obtained by the contraction of the two covariant degrees of θ with the two contravariant degrees of τ .

Proposition A.3. Let M be a compact, connected, and oriented smooth Riemannian n -manifold without boundary isometrically embedded into \mathbb{R}^{n+1} . For the functional $\Phi(h) = \int_M \|h^* g_N - g_M\|^2 \omega_M$ with domain $\text{Diff}(M, N)$, we have that

$$D\Phi(h)(h_*Y) = -4 \int_M g_M(\operatorname{div} B + A : B, Y) \omega_M \quad (38)$$

for all vector fields $Y \in \mathfrak{X}(M)$, where the tensors A and B are as in Definition A.1. Moreover, $h \in \operatorname{Diff}(M, N)$ is a critical point of the functional Φ if and only if

$$\operatorname{div} B + A : B = 0.$$

The latter equation can be rewritten in components as follows:

$$\partial_k B^{km} + \Gamma_{kp}^p B^{km} + \bar{\Gamma}_{kp}^m B^{kp} = 0 \quad (39)$$

for all $m = 1, 2, \dots, n$.

Proof. For given $Y \in \mathfrak{X}(M)$, consider the vector field

$$X = (\beta_{ij} - \alpha_{ij}) \alpha^{ik} \alpha^{jm} Y_m \frac{\partial}{\partial x_k}, \quad (40)$$

where $\alpha = g_M$ and $\beta = h^* g_N$. Although we describe X pointwise using local coordinates, X is a well-defined smooth vector field on M because it is obtained by various contractions of the tensor fields α , β , and Y . It can be verified that the divergence of the vector field X with respect to the Riemannian metric g_M is

$$\operatorname{div}_{g_M} X = \frac{1}{2} G(\beta - \alpha, L_Y \beta) + g_M(\operatorname{div} B + A : B, Y).$$

Because M is without boundary, $\int_M \operatorname{div}_{g_M} X \omega_M = 0$. Using this and the equality

$$D\Phi(h)(h_*Y) = 2 \int_M G(\beta - \alpha, L_Y \beta) \omega_M,$$

we conclude that

$$D\Phi(h)(h_*Y) = -4 \int_M g_M(\operatorname{div} B + A : B, Y) \omega_M$$

as required. \square

Corollary A.4. Let M be a manifold as in Proposition A.3 and $h \in \operatorname{Diff}(M, N)$. If $h^* g_N = R^2 g_M$ for some $R \in \mathbb{R}$, then h is a critical point of the functional Φ .

In particular, let $h_R : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be the radial map defined by $h_R(p) = Rp$ for all $p \in \mathbb{R}^{n+1}$, where $R > 0$. Assume that $N = h_R(M)$ is a rescaled version of the manifold M , and the Riemannian metrics g_M and g_N on the manifolds M and N are inherited from \mathbb{R}^{n+1} . Then every composition $h = f \circ h_R|_M$ of the radial map $h_R|_M$ with an isometry $f \in \operatorname{Diff}(N)$ is a critical point of the functional Φ .

The proof of this corollary is an easy verification of Eq. (39).

Appendix B. Existence and convergence results for evolution operators

In this appendix we state results on existence and convergence of certain evolution operators.

We denote the Euclidean norm of an element $A \in \mathbb{R}^m$, where $m \in \mathbb{N}$, by $|A|$ and the Hilbert space $L^2(0, 1; V^k)$ by \mathcal{H}^k , where the Sobolev space $V^k = W_0^{k,2}(\Omega; \mathbb{R}^{n+1})$ is embedded into $C^r(\bar{\Omega}; \mathbb{R}^{n+1})$ and $r \geq 2$. Recall that Sobolev's theorem guarantees the latter embedding if $k \geq (n+1)/2 + r + 1$. The following lemma is proved in [9].

Lemma B.1 (Dupuis, Grenander, Miller). For every time-dependent vector field $v \in \mathcal{H}^k$ and $t_0 \in [0, 1]$, there exists a function $\phi : [0, 1] \times \Omega \rightarrow \mathbb{R}^{n+1}$ such that $t \mapsto \phi(t, x)$ is the unique absolutely continuous solution of the initial value problem

$$\begin{cases} \frac{dq}{dt} = v(q, t), \\ q(t_0) = x \end{cases} \quad (41)$$

for all $t \in [0, 1]$. Moreover, the function $x \mapsto \phi(t, x)$ is a homeomorphism of Ω .

For every $v \in \mathcal{H}^k$ and $x \in \Omega$, let $F^v(x, t)$ be the solution of the evolution equation $dq/dt = v(q, t)$ with the initial condition $F^v(x, 0) = x$. For a function $f \in C^r(\Omega)$, denote

$$\|f\|_{r,\infty} = \sum_{|\alpha| \leq r} \sup_{x \in \Omega} |D^\alpha f(x)|,$$

where $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ is a multi-index with nonnegative integer components, $|\alpha| = \alpha_1 + \dots + \alpha_{n+1}$, and $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_{n+1}^{\alpha_{n+1}}}$.

More general versions of the following two lemmas are proved in [19, Appendix C].

Lemma B.2 (Troune, Younes). *If $v \in \mathcal{H}^k$ and $F^v : \Omega \times [0, 1] \rightarrow \mathbb{R}^n$ is defined as above, then the function $x \mapsto F^v(x, t)$ is in class $C^r(\Omega)$ and, for all $q \leq r$,*

$$\frac{\partial}{\partial t} D_x^q F^v(x, t) = D_x^q (v(F^v(x, t), t)),$$

where D_x^q denotes the derivative with respect to x of order q . Moreover, there exist positive constants C and C' such that

$$\sup_{t \in [0,1]} \|F^v(\cdot, t)\|_{r,\infty} \leq C e^{C' \|v\|_{\mathcal{H}^k}}$$

for all $v \in \mathcal{H}^k$.

Recall that we say $v^l \rightharpoonup v$ weakly in \mathcal{H}^k as $l \rightarrow \infty$ if $\langle v^l - v, w \rangle \rightarrow 0$ as $l \rightarrow \infty$ for all $w \in \mathcal{H}^k$.

Lemma B.3 (Troune, Younes). *If the sequence $\{v^l\}_{l=1}^\infty \subset \mathcal{H}^k$ converges weakly to $v \in \mathcal{H}^k$ as $l \rightarrow \infty$, then*

$$\sup_{t \in [0,1]} \|F^{v^l}(\cdot, t) - F^v(\cdot, t)\|_{r-1,\infty} \rightarrow 0$$

as $l \rightarrow \infty$. Moreover, there exists a constant $K > 0$ such that

$$\sup_{t \in [0,1]} \|F^{v^l}(\cdot, t)\|_{r,\infty} \leq K \tag{42}$$

for all $l \in \mathbb{N}$.

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